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# Inequalities for $M$ -tensors

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**Abstract**

In this paper, we establish some important properties of  $M$ -tensors. We derive upper and lower bounds for the minimum eigenvalue of  $M$ -tensors, bounds for eigenvalues of  $M$ -tensors except the minimum eigenvalue are also presented; finally, we give the Ky Fan theorem for  $M$ -tensors.

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**Keywords:**  $M$ -tensors; nonnegative tensor; spectral radius; eigenvalues

## 1 Introduction

Eigenvalue problems of higher-order tensors have become an important topic of study in a new applied mathematics branch, numerical multilinear algebra, and they have a wide range of practical applications [1–7].

If there are a complex number  $\lambda$  and a nonzero complex vector  $x$  that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then  $\lambda$  is called the eigenvalue of  $\mathcal{A}$  and  $x$  the eigenvector of  $\mathcal{A}$  associated with  $\lambda$ , where  $\mathcal{A}x^{m-1}$  and  $x^{[m-1]}$  are vectors, whose  $i$ th component is

$$\mathcal{A}x^{m-1} := \left( \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leq i \leq n},$$

$$x^{[m-1]} := (x_i^{m-1})_{1 \leq i \leq n}.$$

This definition was introduced by Qi and Lim [8, 9] where they supposed that  $\mathcal{A}$  is an order  $m$  dimension  $n$  symmetric tensor and  $m$  is even. First, we introduce some results of nonnegative tensors [10–12], which are generalized from nonnegative matrices.

**Definition 1.1** The tensor  $\mathcal{A}$  is called reducible if there exists a nonempty proper index subset  $\mathbb{J} \subset \{1, 2, \dots, n\}$  such that  $a_{i_1 i_2 \dots i_m} = 0$ ,  $\forall i_1 \in \mathbb{J}, \forall i_2, \dots, i_m \notin \mathbb{J}$ . If  $\mathcal{A}$  is not reducible, then we call  $\mathcal{A}$  to be irreducible.

Let  $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}$ , where  $|\lambda|$  denotes the modulus of  $\lambda$ . We call  $\rho(\mathcal{A})$  the spectral radius of tensor  $\mathcal{A}$ .

**Theorem 1.2** *If  $\mathcal{A}$  is irreducible and nonnegative, then there exists a number  $\rho(\mathcal{A}) > 0$  and a vector  $x_0 > 0$  such that  $\mathcal{A}x_0^{m-1} = \rho(\mathcal{A})x_0^{[m-1]}$ . Moreover, if  $\lambda$  is an eigenvalue with a nonnegative eigenvector, then  $\lambda = \rho(\mathcal{A})$ . If  $\lambda$  is an eigenvalue of  $\mathcal{A}$ , then  $|\lambda| \leq \rho(\mathcal{A})$ .*

The authors in [13, 14] extended the notion of  $M$ -matrices to higher-order tensors and introduced the definition of an  $M$ -tensor.

**Definition 1.3** Let  $\mathcal{A}$  be an  $m$ -order and  $n$ -dimensional tensor.  $\mathcal{A}$  is called an  $M$ -tensor if there exist a nonnegative tensor  $\mathcal{B}$  and a real number  $c > \rho(\mathcal{B})$ , where  $\mathcal{B}$  is the spectral radius of  $\mathcal{B}$ , such that

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

**Theorem 1.4** *Let  $\mathcal{A}$  be an  $M$ -tensor and denote by  $\tau(\mathcal{A})$  the minimal value of the real part of all eigenvalues of  $\mathcal{A}$ . Then  $\tau(\mathcal{A}) > 0$  is an eigenvalue of  $\mathcal{A}$  with a nonnegative eigenvector. Moreover, there exist a nonnegative tensor  $\mathcal{B}$  and a real number  $c > \rho(\mathcal{B})$  such that  $\mathcal{A} = c\mathcal{I} - \mathcal{B}$ . If  $\mathcal{A}$  is irreducible, then  $\tau(\mathcal{A})$  is the unique eigenvalue with a positive eigenvector.*

In this paper, let  $N = \{1, 2, \dots, n\}$ , we define the  $i$ th row sum of  $\mathcal{A}$  as  $R_i(\mathcal{A}) = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m}$ , and denote the largest and the smallest row sums of  $\mathcal{A}$  by

$$R_{\max}(\mathcal{A}) = \max_{i=1, \dots, n} R_i(\mathcal{A}), \quad R_{\min}(\mathcal{A}) = \min_{i=1, \dots, n} R_i(\mathcal{A}).$$

Furthermore, a real tensor of order  $m$  dimension  $n$  is called the unit tensor, if its entries are  $\delta_{i_1 \dots i_m}$  for  $i_1, \dots, i_m \in N$ , where

$$\delta_{i_1 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

And we define  $\sigma(\mathcal{A})$  as the set of all the eigenvalues of  $\mathcal{A}$  and

$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2 \dots i_m}=0} |a_{ii_2 \dots i_m}|, \quad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2 \dots i_m}=0, \\ \delta_{ji_2 \dots i_m}=0}} |a_{ii_2 \dots i_m}| = r_i(\mathcal{A}) - |a_{ij \dots j}|.$$

In this paper, we continue this research on the eigenvalue problems for tensors. In Section 2, some bounds for the minimum eigenvalue of  $M$ -tensors are obtained, and proved to be tighter than those in Theorem 1.1 in [15]. In Section 3, some bounds for eigenvalues of  $M$ -tensors except the minimum eigenvalue are given. Moreover, the Ky Fan theorem for  $M$ -tensors is presented in Section 4.

## 2 Bounds for the minimum eigenvalue of $M$ -tensors

**Theorem 2.1** *Let  $\mathcal{A}$  be an irreducible  $M$ -tensor. Then*

$$\tau(\mathcal{A}) \leq \min\{a_{i \dots i}\}, \tag{1}$$

$$R_{\min}(\mathcal{A}) \leq \tau(\mathcal{A}) \leq R_{\max}(\mathcal{A}). \tag{2}$$

*Proof* Let  $x > 0$  be an eigenvector of  $\mathcal{A}$  corresponding to  $\tau(\mathcal{A})$ , i.e.,  $\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}$ . For each  $i \in N$ , we can get

$$(a_{i\dots i} - \tau(\mathcal{A}))x_i^{m-1} = - \sum_{\delta_{i i_2 \dots i_m} = 0} a_{i i_2 \dots i_m} x_{i_2} \cdots x_{i_m} \geq 0,$$

then

$$\tau(\mathcal{A}) \leq \min\{a_{i\dots i}\}.$$

Assume that  $x_s$  is the smallest component of  $x$ ,

$$(a_{s\dots s} - \tau(\mathcal{A}))x_s^{m-1} = - \sum_{\delta_{s i_2 \dots i_m} = 0} a_{s i_2 \dots i_m} x_{i_2} \cdots x_{i_m} \geq 0.$$

That is,

$$\tau(\mathcal{A}) \leq \sum_{\delta_{s i_2 \dots i_m} = 0} a_{s i_2 \dots i_m} + a_{s\dots s},$$

so

$$\tau(\mathcal{A}) \leq R_{\max}(\mathcal{A}).$$

Similarly, if we assume that  $x_t = \{\max x_i, i \in N\}$ , then we can get

$$\tau(\mathcal{A}) \geq \sum_{\delta_{t i_2 \dots i_m} = 0} a_{t i_2 \dots i_m} + a_{t\dots t} \geq R_{\min}(\mathcal{A}).$$

Thus, we complete the proof.  $\square$

**Theorem 2.2** Let  $\mathcal{A}$  be an irreducible  $M$ -tensor. Then

$$\begin{aligned} \min_{i,j \in N, j \neq i} \frac{1}{2} \{a_{i\dots i} + a_{j\dots j} - r_i^j(\mathcal{A}) - \Delta_{ij}^{\frac{1}{2}}(\mathcal{A})\} \\ \leq \tau(\mathcal{A}) \leq \max_{i,j \in N, j \neq i} \frac{1}{2} \{a_{i\dots i} + a_{j\dots j} - r_i^j(\mathcal{A}) - \Delta_{ij}^{\frac{1}{2}}(\mathcal{A})\}, \end{aligned} \quad (3)$$

where

$$\Delta_{ij}(\mathcal{A}) = (a_{i\dots i} - a_{j\dots j} + r_i^j(\mathcal{A}))^2 - 4a_{ij\dots j}r_j(\mathcal{A}).$$

*Proof* Because  $\tau(\mathcal{A})$  is an eigenvalue of  $\mathcal{A}$ , from Theorem 2.1 in [15], there are  $i, j \in N, j \neq i$ , such that

$$(|\tau(\mathcal{A}) - a_{i\dots i}| - r_i^j(\mathcal{A}))|\tau(\mathcal{A}) - a_{j\dots j}| \leq |a_{ij\dots j}r_j(\mathcal{A})|.$$

From Theorem 2.1, we can get

$$(a_{i\dots i} - \tau(\mathcal{A}) - r_i^j(\mathcal{A}))(a_{j\dots j} - \tau(\mathcal{A})) \leq -a_{ij\dots j}r_j(\mathcal{A}),$$

equivalently,

$$\tau(\mathcal{A})^2 - (a_{i\dots i} + a_{j\dots j} - r_i^j(\mathcal{A}))\tau(\mathcal{A}) + a_{j\dots j}(a_{i\dots i} - r_i^j(\mathcal{A})) + a_{ij\dots j}r_j(\mathcal{A}) \leq 0.$$

Then, solving for  $\tau(\mathcal{A})$ ,

$$\tau(\mathcal{A}) \geq \frac{1}{2} \{a_{i\dots i} + a_{j\dots j} - r_i^j(\mathcal{A}) - \Delta_{ij}^{\frac{1}{2}}(\mathcal{A})\} \geq \min_{i,j \in N, j \neq i} \frac{1}{2} \{a_{i\dots i} + a_{j\dots j} - r_i^j(\mathcal{A}) - \Delta_{ij}^{\frac{1}{2}}(\mathcal{A})\}.$$

Let  $x > 0$  be an eigenvector of  $\mathcal{A}$  corresponding to  $\tau(\mathcal{A})$ , i.e.,  $\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}$ ,  $x_s$  is the smallest component of  $x$ . For each  $s, t \in N$ ,  $s \neq t$ , we can get

$$(a_{t\dots t} - \tau(\mathcal{A}))x_t^{m-1} = - \sum_{\delta_{ti_2\dots i_m}=0} a_{ti_2\dots i_m}x_{i_2} \cdots x_{i_m} \geq r_t(\mathcal{A})x_s^{m-1}, \quad (4)$$

$$(a_{s\dots s} - \tau(\mathcal{A}))x_s^{m-1} = - \sum_{\substack{\delta_{ti_2\dots i_m}=0, \\ \delta_{si_2\dots i_m}=0}} a_{ti_2\dots i_m}x_{i_2} \cdots x_{i_m} - a_{st\dots t}x_t^{m-1} \geq r_t^s(\mathcal{A})x_s^{m-1} - a_{st\dots t}x_t^{m-1},$$

$$(a_{s\dots s} - \tau(\mathcal{A}) - r_t^s(\mathcal{A}))x_s^{m-1} \geq -a_{st\dots t}x_t^{m-1}. \quad (5)$$

Multiplying equations (4) and (5), we get

$$(a_{t\dots t} - \tau(\mathcal{A}))(a_{s\dots s} - \tau(\mathcal{A}) - r_t^s(\mathcal{A})) \geq -a_{st\dots t}r_t(\mathcal{A}).$$

Then, solving for  $\tau(\mathcal{A})$ ,

$$\tau(\mathcal{A}) \leq \frac{1}{2} \{a_{t\dots t} + a_{s\dots s} - r_t^s(\mathcal{A}) - \Delta_{t,s}^{\frac{1}{2}}(\mathcal{A})\} \leq \max_{i,j \in N, j \neq i} \frac{1}{2} \{a_{i\dots i} + a_{j\dots j} - r_i^j(\mathcal{A}) - \Delta_{ij}^{\frac{1}{2}}(\mathcal{A})\}.$$

Thus, we complete the proof.  $\square$

We now show that the bounds in Theorem 2.2 are tight and sharper than those in Theorem 1.1 in [15] by the following example. Consider the  $M$ -tensor  $\mathcal{A} = (a_{ijkl})$  of order 4 dimension 2 with entries defined as follows:

$$a_{1111} = 3, \quad a_{1222} = -1,$$

$$a_{2111} = -2, \quad a_{2222} = 2,$$

other  $a_{ijkl} = 0$ . By Theorem 1.1 in [15], we have

$$-2 \leq \tau(\mathcal{A}) \leq 4.$$

By Theorem 2.1, we have

$$0 \leq \tau(\mathcal{A}) \leq 2.$$

By Theorem 2.2, we have

$$\frac{1}{2}(5 - \sqrt{17}) \leq \tau(\mathcal{A}) \leq \frac{1}{2}(5 - \sqrt{5}).$$

In fact,  $\tau(\mathcal{A}) = 1$ . Hence, the bounds in Theorem 2.2 are tight and sharper than those in Theorem 1.1 in [15].

### 3 Bounds for eigenvalues of $M$ -tensors except the minimum eigenvalue

In this section, we introduce the stochastic  $M$ -tensor, which is a generalization of the non-negative stochastic tensor.

**Definition 3.1** An  $M$ -tensor  $\mathcal{A}$  of order  $m$  dimension  $n$  is called stochastic provided

$$R_i(\mathcal{A}) = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} \equiv 1, \quad i = 1, \dots, n.$$

Obviously, when  $\mathcal{A}$  is a stochastic  $M$ -tensor, 1 is the minimum eigenvalue of  $\mathcal{A}$  and  $e$  is an eigenvector corresponding to 1, where  $e$  is an all-ones vector.

**Theorem 3.2** Let  $\mathcal{A}$  be an order  $m$  dimension  $n$  irreducible  $M$ -tensor. Then there exists a diagonal matrix  $D$  with positive main diagonal entries such that

$$\tau(\mathcal{A}) \cdot \mathcal{B} = \mathcal{A} \cdot D^{(1-m)} \cdot \overbrace{D \cdot \dots \cdot D}^{m-1},$$

where  $\mathcal{B}$  is a stochastic irreducible  $M$ -tensor. Furthermore,  $\mathcal{B}$  is unique, and the diagonal entries of  $D$  are exactly the components of the unique positive eigenvector corresponding to  $\tau(\mathcal{A})$ .

*Proof* Let  $x$  be the unique positive eigenvector corresponding to  $\tau(\mathcal{A})$ , i.e.,

$$\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}.$$

Let  $D$  be the diagonal matrix such that its diagonal entries are components of  $x$ , let us check the tensor  $\mathcal{C} = \mathcal{A} \cdot D^{(1-m)} \cdot D \cdot \dots \cdot D$ . It is clear that for  $i = 1, 2, \dots, n$ ,

$$\sum_{i_2, \dots, i_m=1}^n C_{ii_2 \dots i_m} = (\mathcal{C}e^{m-1})_i = (\mathcal{A} \cdot D^{(1-m)} \cdot \overbrace{D \cdot \dots \cdot D}^{m-1} e^{m-1})_i = \tau(\mathcal{A}).$$

Hence  $\mathcal{B} = \mathcal{C}/\tau(\mathcal{A})$  is the desired stochastic  $M$ -tensor. Since the positive eigenvector is unique, then  $\mathcal{B}$  is unique, and the diagonal entries of  $D$  are exactly the components of the unique positive eigenvector corresponding to  $\tau(\mathcal{A})$ .  $\square$

**Theorem 3.3** Let  $\mathcal{A}$  be an order  $m$  dimension  $n$  stochastic irreducible nonnegative tensor,  $\omega = \min a_{i \dots i}$ ,  $\lambda \in \sigma(\mathcal{A})$ . Then

$$|\lambda - \omega| \leq 1 - \omega.$$

*Proof* Let  $\lambda$  be an eigenvalue of the stochastic irreducible nonnegative tensor  $\mathcal{A}$ ,  $x$  is the eigenvector corresponding to  $\lambda$ , i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

Assume that  $0 < |x_s| = \max_i |x_i|$ , then we can get

$$(\lambda - a_{s \dots s})x_s^{m-1} = \sum_{\delta_{si_2 \dots i_m} = 0} a_{si_2 \dots i_m} x_{i_2} \cdots x_{i_m}.$$

Then

$$|\lambda - a_{s \dots s}| \leq \sum_{\delta_{si_2 \dots i_m} = 0} a_{si_2 \dots i_m} = r_s(\mathcal{A}) = 1 - a_{s \dots s},$$

and therefore,

$$\begin{aligned} |\lambda - \omega| &\leq |\lambda - a_{s \dots s} + a_{s \dots s} - \omega| \\ &\leq |\lambda - a_{s \dots s}| + |a_{s \dots s} - \omega| \\ &\leq (1 - a_{s \dots s}) + (a_{s \dots s} - \omega) \\ &= 1 - \omega. \end{aligned} \tag{6}$$

Thus, we complete the proof.  $\square$

**Theorem 3.4** Let  $\mathcal{A}$  be an order  $m$  dimension  $n$  irreducible  $M$ -tensor,  $\Omega = \max a_{i \dots i}$ ,  $\lambda \in \sigma(\mathcal{A})$ . Then

$$|\Omega - \lambda| \leq \Omega - \tau(\mathcal{A}).$$

*Proof* From Theorem 3.2, we may evidently take  $\tau(\mathcal{A}) = 1$ , and after performing a similarity transformation with a positive diagonal matrix, we may assume that  $\mathcal{A}$  is stochastic. Then, for  $\theta \in (0, 1)$ , the matrix  $\mathcal{A}(\theta) = (1 + \theta)\mathcal{I} - \theta\mathcal{A}$  is irreducible nonnegative stochastic, by Theorem 3.3, if  $\lambda(\theta) \in \sigma(\mathcal{A}(\theta))$ ,  $\omega(\theta) = \min a_{i \dots i}(\theta)$ , we can get

$$|\lambda(\theta) - \omega(\theta)| \leq 1 - \omega(\theta).$$

That is,

$$|1 + \theta - \theta\lambda - (1 + \theta - \theta \max a_{i \dots i})| \leq 1 - (1 + \theta - \theta \max a_{i \dots i}).$$

Then

$$|\Omega - \lambda| \leq \Omega - 1.$$

Transforming back to  $\mathcal{A}$ , we get

$$|\Omega - \lambda| \leq \Omega - \tau(\mathcal{A}).$$

Thus, we complete the proof.  $\square$

#### 4 Ky Fan theorem for $M$ -tensors

In this section we give the Ky Fan theorem for  $M$ -tensors. Denote by  $\mathbb{Z}$  the set of  $m$ -order and  $n$ -dimensional real tensors whose off-diagonal entries are nonpositive.

**Theorem 4.1** *Let  $\mathcal{A}, \mathcal{B} \in \mathbb{Z}$ , assume that  $\mathcal{A}$  is an  $M$ -tensor and  $\mathcal{B} \geq \mathcal{A}$ . Then  $\mathcal{B}$  is an  $M$ -tensor, and*

$$\tau(\mathcal{A}) \leq \tau(\mathcal{B}).$$

*Proof* If  $x > 0$ , from assume that  $\mathcal{A}$  is an  $M$ -tensor and condition (D4) in [14], we know

$$\mathcal{A}x^{m-1} > 0.$$

Because  $\mathcal{B} \geq \mathcal{A}$ , we can get

$$\mathcal{B}x^{m-1} \geq \mathcal{A}x^{m-1} > 0,$$

then  $\mathcal{B}$  is an  $M$ -tensor.

Let  $a = \max_{1 \leq i \leq n} \mathcal{B}_{i \dots i}$ , from Theorem 3.1 and Corollary 3.2 in [13], assume that

$$\mathcal{B} = a\mathcal{I} - \mathcal{C}_B, \quad \mathcal{A} = a\mathcal{I} - \mathcal{C}_A,$$

where  $\mathcal{C}_A, \mathcal{C}_B$  are nonnegative tensors.

Because  $\mathcal{A}, \mathcal{B} \in \mathbb{Z}$  and  $\mathcal{B} \geq \mathcal{A}$ , then we can get

$$\mathcal{C}_A \geq \mathcal{C}_B.$$

From Lemma 3.5 in [12], we can get

$$\rho(\mathcal{C}_A) \geq \rho(\mathcal{C}_B).$$

Therefore,

$$\tau(\mathcal{A}) \leq \tau(\mathcal{B}).$$

Thus, we complete the proof. □

**Theorem 4.2** *Let  $\mathcal{A}, \mathcal{B}$  be of order  $m$  dimension  $n$ , suppose that  $\mathcal{B}$  is an  $M$ -tensor and  $|b_{i_1 \dots i_m}| \geq |a_{i_1 \dots i_m}|$  for all  $i_1 \neq \dots \neq i_m$ . Then, for any eigenvalue  $\lambda$  of  $\mathcal{A}$ , there exists  $i \in 1, \dots, n$  such that  $|\lambda - a_{i \dots i}| \leq b_{i \dots i} - \tau(\mathcal{B})$ .*

*Proof* We first suppose that  $\mathcal{B}$  is an  $M$ -tensor,  $\tau(\mathcal{B})$  is an eigenvalue of  $\mathcal{B}$  with a positive corresponding eigenvector  $v$ . Denote

$$W = \text{diag}(v_1, \dots, v_n),$$

where  $v_i$  is the  $i$ th component of  $v$ . Let

$$\mathcal{C} = \mathcal{A} \cdot W^{1-m} \overbrace{W \cdots W}^{[m-1]}$$

and let  $\lambda$  be an eigenvalue of  $\mathcal{A}$  with  $x$ , a corresponding eigenvector, i.e.,  $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$ . Then, as in the proof of Theorem 4.1 in [12], we have

$$\mathcal{C}(W^{-1}x)^{m-1} = \lambda(W^{-1}x)^{m-1}.$$

By the definition of  $\mathcal{C}$ , we have  $c_{i \dots i} = a_{i \dots i}$ ,  $i = 1, \dots, n$ . Applying the first conclusion of Theorem 6 of [8], we can get

$$\begin{aligned} |\lambda - c_{i \dots i}| &\leq \sum_{\delta_{i i_2 \dots i_m} = 0} |c_{i i_2 \dots i_m}| \\ &= v_i^{1-m} \sum |a_{i i_2 \dots i_m}| v_{i_2} \cdots v_{i_m} \\ &\leq v_i^{1-m} \sum |b_{i i_2 \dots i_m}| v_{i_2} \cdots v_{i_m} \\ &= v_i^{1-m} \left( b_{i \dots i} v^{m-1} - \sum_{i_1, \dots, i_m=1} b_{i i_2 \dots i_m} v_{i_2} \cdots v_{i_m} \right) \\ &= b_{i \dots i} - \tau(\mathcal{B}). \end{aligned} \quad (7)$$

Thus, we complete the proof.  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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